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ON THE CALCULATION OF WEIGHTED AVERAGE FIBER LENGTH IN PAPER

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ABSTRACT

The physical properties of pulp and paper, when related to fiber length, depend more on a weighted average length by fiber weight than on a mean length based only on fiber number. However, a weighted average length by true weight is often difficult or impossible to obtain, and one must compromisingly settle for a functional estimate of this statistic based only on fiber length. The present report reexamines the early assumptions and commentary on this subject in order to clarify the estimated statistic and, using assumptions somewhat different from those adopted by earlier workers, rederives and revalidates the formula used for its computation. The limitations for the use of this estimated, weighted average length are also discussed.

INTRODUCTION

"What is the average fiber length of our pulp?" Although this is a frequent question of any papermaker, such thoughts are especially relevant in recent times due to the increasing use of various secondary fibers and/or wood residuals. Albeit fiber length is not the sole important parameter of fiber morphology determining the physical properties of pulp and paper, it is among the foremost, and a statistic based on fiber length is often critical to the successful prediction of some paper characteristics (Dinwoodie 1965, Watson and Dadswell 1961, Clark 1975). Unfortunately, as has been explained in earlier literature, the meaning of the expression "average fiber length," or more specifically, "average paper fiber length," connotes different things to different people (Clark 1942). The objectives of the present report are twofold, (1) to reiterate some of these early but extremely astute observations on the interpretation of paper fiber length, and (2) to expand upon the derivation of a formula commonly used in the paper industry to compute the weighted average fiber length by weight by measuring only fiber length (see Clark 1942, 1962).

WHAT IS A "PAPER FIBER?"

Clark (1942, 1962) reviewed the various opinions on exactly what elements in a sheet of paper should or should not be considered a "paper fiber." An early point of contention was the question of what minimum length should be regarded as the lower limit to distinguish "fibers" from "debris." There was and still is good reason to consider 0.1 mm as the lower limit since this is the approximate thickness of the average paper. Any material shorter than this cannot sensibly be held to contribute to fiber length *per se* (Clark 1962). Consequently, for routine "fiber" analysis (T 401 os-74), the analyst should logically count whole fibers as well as broken fibers or fragments in the $\geq$ 0.1-mm range. It is customary to delete from the count, however, other very narrow fragments and/or nonfibrous cell types such as parenchyma, vessel elements, or ray tracheids.

WHAT IS AN "AVERAGE FIBER LENGTH?"

The simple "numerical average" or "arithmetic mean," $\bar{L}_N$, of a sample of paper "fibers" may be defined as the sum of all the lengths divided by the total number of "fibers" (Wine 1964). This statistic is also equivalent to that obtained (see Fig. 1) by summing, over a series of length classes, the products of the frequency of fibers in a given class (or the percent frequency) and the mean length in that class, and dividing this sum by the total frequency (or percent frequency, 100%) (Wine 1964). If one employs the simple arithmetic or numerical average to describe a sample of nonuniformly long paper fibers (i.e., the typical paper), he may arrive at a surprising, as well as actually useless, result. Clark (1962) pointed out that for a sample of 5 unbroken fibers, each 5 mm long, the arithmetic mean length is clearly 5 mm. However, if one of these fibers is cut uniformly into 46 equal fragments of 0.11 mm, the sample mean is reduced to only 0.5 mm. If the same fiber were instead divided into 246 equal lengths, the mean length drops to 0.1 mm, or a reduction in original mean length by 98% by altering only 20% of the sample. Obviously, as Clark (1962) surmised, such a statistic is not meaningful as an effective measure of "functional" mean length, and a better term is required. The arithmetic mean gives too much importance to the shorter fibers and fragments while resultant paper properties depend more on the longer fibers (Clark 1942).

	Fiber Length Classes	Fiber Frequency	% Frequency	Mean Length, mm
(shortest)	1	f_1	F_1	$\bar{L}_1$
	2	f_2	F_2	$\bar{L}_2$
	3	f_3	F_3	$\bar{L}_3$
	⋮	⋮		
(longest)	n	f_n	$F_n = \frac{F_n}{N} \cdot 100$	$\bar{L}_n$
	Totals	N	100	

$$\bar{L}_N = \frac{\sum_{i=1}^{n} f_i \bar{L}_i}{\sum_{i=1}^{n} f_i} = \frac{\sum_{i=1}^{n} F_i \bar{L}_i}{100} \qquad (1)$$

Figure 1. Calculation of the Simple Numerical or Arithmetic Mean Fiber Length, $\bar{L}_N$, of a Sample of N Fibers

A method by which one can attribute or weight the relative importance of single measurements in a given sample is to calculate a "weighted average" (Wine 1964). If, for example, in the computation of a mean fiber length, one wishes to give more emphasis to the longer fibers, he must decide on an appropriate characteristic or "weighting factor" for these fibers and one that is also readily determined. An obvious weighting factor is the weight of the fibers themselves, since the longer fibers probably weigh more than the shorter ones and certainly more than small fragments. A true weighted mean length according to weight ($\overline{L}_W$) can then be computed as

$$\overline{L}_W = \frac{\sum_{i=1}^{n} w_i L_i}{\sum w_i} \qquad (2)$$

where

w_i = individual fiber weights

L_i = individual fiber lengths

n = number of fibers counted (sample size)

In practice, calculation of the foregoing statistic is clearly not feasible, and an approximation or compromise is necessary. The latter may be obtained by first physically classifying a sample into several ranges of length, measuring the arithmetic mean of each range, taking the actual weight of each range, and then computing $\overline{L}_W$ of the whole (T 233 su-64, Clark 1962). The approximation here is quite good since the arithmetic mean of each range of lengths is close to the weighted mean of that range (as the lengths within each classified range are relatively uniform).

For a mixture of pulps of different fiber length distributions, it can be shown that the physical properties of the mixture depend upon the weighted average fiber length by weight, as calculated above, and not by number (Clark 1942).

Because of this dependence, together with the fact that pulps are always blended by weight, reference to the mean fiber length of any pulp should be to the weighted average by true weight inasmuch as possible (Clark 1942).

THE WEIGHTED AVERAGE LENGTH IN PRACTICE

For routine applications on numerous samples, calculation of $\bar{L}_W$ by the foregoing procedures is laborious. Furthermore, analysis of very small samples — those which preclude a fiber classification — can be impossible. A more convenient approach, if valid, would be to obtain $\bar{L}_W$ indirectly by measuring only fiber lengths. Obviously, such an accomplishment or "shortcut" would necessarily dictate that a relationship between fiber weight and length be known and entered into the computation.

Various schemes for a "shortcut" to $\bar{L}_W$ have been published by Clark (1942, 1962) and in TAPPI T 232 su-68. These methods assume that for each range of fiber lengths, the weight per unit length of these fibers — that is, the coarseness or decigrex (T 234 su-67) — is known, or at least factors proportional to coarseness are known. Such factors can be obtained experimentally for the individual length classes of a given pulp sample (T 234 su-67, Britt 1966, Ranger 1961), but this is very time consuming and the sample must be sufficiently large to permit classification or handsheet formation. Alternatively, one can assume that the coarseness factors of a classified reference pulp (T 234 su-67) do not vary greatly for other pulp sources. However, this assumption is very weak. Both coarseness and fiber length vary significantly among hardwoods, softwoods, juvenile wood, and mature wood, and there seems to be no data available to substantiate that their interrelationship is easily predicted for complex samples. Furthermore, there is evidence to show that coarseness of even a single species also varies with pulp yield (Einspahr and Hankey 1977).

Despite the aforementioned limitations on an assumption that a relationship between fiber length and coarseness is predictable, evidence indicates there is indeed <u>some type</u> of direct relationship (at least on an individual fiber basis or for similar fibers) between fiber length and fiber weight. And perhaps while an exact mathematical description between length and weight may not be possible for complex pulps, for practical applications or decisions concerning paper properties, the very general assumption that a heavier fiber is a longer fiber can be statistically supported.

Based on the foregoing premise, Clark (1942) devised a statistic to provide $\overline{L}_W$ by measuring only fiber length. The formula is well suited to the measuring-wheel or similar approaches for projected fiber images (see Clark 1962), and the calculation algorithm produces the weighted average length as

$$\overline{L}_W = \frac{\sum_{i=1}^{n} \left(F_i\right) \overline{L}_i^{\,2}}{\sum_{i=1}^{n} \left(F_i\right) \overline{L}_i} \qquad (3)$$

where

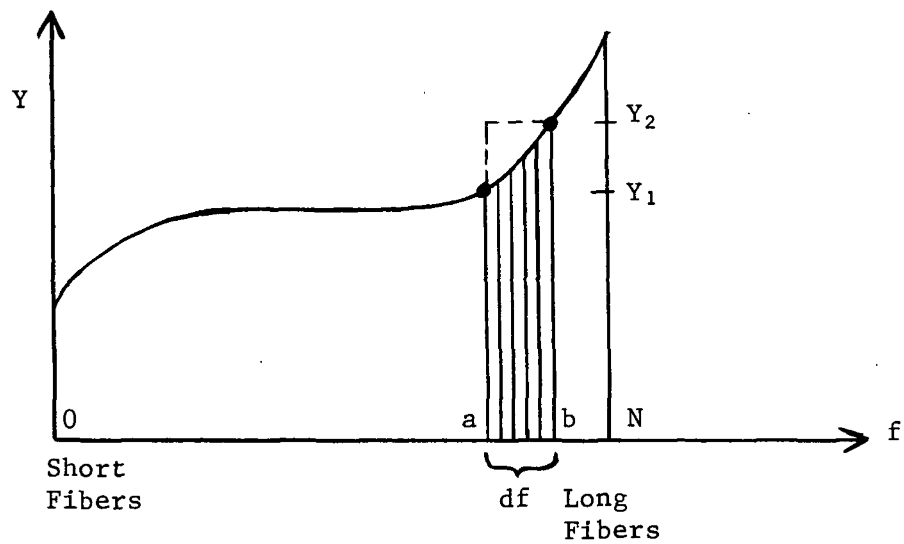
F_i = class frequency, %

$\overline{L}_i$ = class average fiber length

n = number of length classes

This formula is widely used throughout the paper industry in the U.S. (Isenberg 1967) as well as abroad (Unger 1975).

If one examines Equation (3), he might first get the impression that the statistic is weighted according to length, judging from the squared length term in the numerator. Nevertheless, the statistic is derived according to weight under the aforementioned assumption that fiber weight is proportional to fiber length (Clark 1942, 1962). The mathematical development of this formula is, however, at least to the authors' knowledge, not to be found specifically as such in the literature and is apparently extractable only from the hypothetical operations of Clark (1942) on a population of fibers of assumed equal coarseness (Clark 1976). This hypothetical treatment of a population of fibers arranged side-by-side in order of increasing length is given below and illustrated in Fig. 2.



f = fiber frequency

Y = fiber length

df = differential frequency of fibers in any subfraction (assuming fibers of equal coarseness)

$\bar{Y}$ = arithmetic average of fiber length in df

= $(Y_1 + Y_2)/2$

N = total population of fibers

Figure 2. Diagrammatic Arrangement of Fibers in a Paper Sample, Showing Side-by-Side Placement from Shortest to Longest Fibers. All Fibers Are Assumed to be of Equal Coarseness (from Clark 1942)

If the number of fibers in each subfraction of a sample is df, the weighted average length in df according to projected fiber area, $\overline{L}_a$ (since fiber weight is not known), is computed as follows:

$$\overline{L}_a = \frac{(\overline{Y}\cdot df)\overline{Y}}{\overline{Y}\cdot df} = \frac{\text{(projected fiber area) av. length}}{\text{projected fiber area}} \qquad (4)$$

The weighted average length of the whole sample, $\overline{L}_A$, can then be computed as the sum of the areas of each subfraction from f=0 to f=N, or

$$\overline{L}_A \cong \frac{\int_o^N \overline{Y}^2 df}{\int_o^N \overline{Y} df} \qquad (5)$$

If we accept the assumption that df is representative of the fiber frequency in each subfraction of fibers (Clark 1942), then Equation (5) is essentially equivalent to Equation (3) and, assuming equal fiber coarseness, $\overline{L}_A = \overline{L}_W$.

In the literature, there is apparently no other derivation of Equation (3) other than the foregoing, which was devised by Clark (1942). Even in Clark's treatment of this subject, however, the assumption of equal fiber coarseness or that fiber coarseness or weight is proportional to length is not shown directly in the calculation of $\overline{L}_W$. The following mathematical treatment approaches and utilizes this basic assumption more directly and provides a more straightforward algorithm to yield the same $\overline{L}_W$ as defined by Clark in Equation (3).

In Equation (2) for $\overline{L}_W$, the true weighted mean fiber length by individual fiber weight, it was pointed out that $\overline{L}_W$ is also equivalent to

$$\frac{\sum_{i=1}^{n} W_i \overline{L}_i}{\sum_{i=1}^{n} W_i} \quad \text{or} \quad \frac{\sum_{i=1}^{n} W_i \,\%\, \overline{L}_i}{\sum_{i=1}^{n} W_i \,\%} \qquad (6)$$

where n = number of length classes, W_i = the weight of a particular fraction or range of fiber lengths, and $\overline{L}_i$ = the arithmetic mean length of that same fraction. We can also reason that a functional approximation for W_i would be $W_i = N_i\overline{w}_i$, where N_i = number of fibers in any fraction and $\overline{w}_i$ = average weight per fiber in the same fraction. Thus, for each fraction,

$$W_i\ \% = N_i\ \% \cdot \overline{w}_i \tag{7}$$

If we then assume that the average weight per fiber in the fraction is proportional to the average length of the same fibers (i.e., constant fiber coarseness), then

$$\overline{w}_i \propto \overline{L}_i, \quad \overline{w}_i = a\overline{L}_i \tag{8}$$

$$a = \text{constant}$$

In Equation (6) where $W_i\ \% = N_i\ \%\ \overline{w}_i$, substituting for $\overline{w}_i$, we obtain

$$W_i\ \% = (N_i\ \%)(a\overline{L}_i) \tag{9}$$

Consequently, upon substituting for $W_i\ \%$ in Equation (6) we obtain

$$\overline{L}_W = \frac{\sum_{i=1}^{n}[(N_i\ \%)(a\overline{L}_i)]\overline{L}_i}{\sum_{i=1}^{n}[(N_i\ \%)(a\overline{L}_i)]} = \frac{\sum N_i\ \%\ \overline{L}_i^2}{\sum N_i\ \%\ \overline{L}_i}$$

which is equivalent to Equation (3).

The foregoing discussion has dealt primarily with the parameter of paper fiber length. However, to gain a more comprehensive understanding of closely related factors such as fiber coarseness, and another interrelated pulp characteristic employed for quantitative fiber analysis — weight factor — one should refer to previous work carried out at The Institute of Paper Chemistry under Project 3033. The latter research effort was concerned specifically with improved weight factors for fiber analysis.

SUMMARY

The measurement of the weighted average fiber length, $\overline{L}_w$, by actual weight of individual fibers or of fractions of different fiber lengths for numerous and/or small populations of paper fibers, is often either impractical or impossible. Consequently, when $\overline{L}_w$ is desired under these circumstances, an assumption of a direct relationship between fiber weight and length must be made and an estimate of $\overline{L}_w$ computed from data only on fiber length. While not strictly valid for complex pulps, this assumption does permit calculation of a functional measure of paper fiber length, which by necessity must emphasize the longer fibers.

An explicit mathematical derivation of $\overline{L}_w$ based only on fiber length and the compromises involved are not to be found in the literature. The early hypothetical treatment of the subject and derivation of a functional statistic by Clark (1942), however, has served for many years as the sole basis on which $\overline{L}_w$ is computed. We have reexamined Clark's approach to the problem and have obtained in a more straightforward fashion, but with somewhat different assumptions, the same equation. We hope that the present report will help clarify the premise on which $\overline{L}_w$ is based as well as the limitations or compromises surrounding its computation.

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